UDC 536.2

SOLUTION OF THE HEAT-CONDUCTION EQUATION WITH DISCONTINUOUS PARAMETERS AND ITS APPLICATION TO THE PROBLEM OF ELECTRIC CONTACTS

E. I. Kim, V. T. Omel'chenko, and S. N. Kharin

Inzhenerno-Fizicheskii Zhurnal, Vol. 8, No. 6, pp. 761-767, 1965

The inhomogeneous heat-conduction equation with moving boundaries and a discontinuous coefficient of thermal diffusivity is solved in closed form. The solution is applied to the calculation of the amount of material transferred from one electrode to the other when a dc circuit is opened.

Formulation of the Problem

It is required to solve the problem

$$\frac{\partial u_1}{\partial t} = a_1^2 \frac{\partial^2 u_1}{\partial x^2} + b_1, \quad -\alpha_1\sqrt{t} < x < 0, \tag{1}$$

$$\frac{\partial u_2}{\partial t} = a_2^2 \frac{\partial^2 u_2}{\partial x^2} + b_2, \quad 0 < x < \alpha_2\sqrt{t} \tag{2}$$

with the boundary conditions

$$u_1(-\alpha_1\sqrt{t},\ t) = u_{n1}, \tag{3}$$

$$u_2(\alpha_2\sqrt{t},\ t) = u_{n2} \tag{4}$$

and the coupling conditions

$$u_1(0,\ t) = u_2(0,\ t), \tag{5}$$

$$\lambda_1 \frac{\partial u_1(0,\ t)}{\partial x} = \lambda_2 \frac{\partial u_2(0,\ t)}{\partial x}, \tag{6}$$

where a_i^2, b_i, α_i, u_{ni}, λ_i are unknown constants. When $u_{n1} \neq u_{n2}$, the solution has a discontinuity at the point (0, 0). Therefore we shall require only that the solution be bounded in the neighborhood of (0, 0).

We shall seek a solution of the form

$$u_1(x,\ t) = b_1 t + A_1 \operatorname{erf}\left(\frac{x}{2a_1\sqrt{t}}\right) + B_1\left(t + \frac{x^2}{2a_1^2}\right) + C_1 \int_0^t \frac{\sqrt{\tau}}{(t-\tau)^{1/2}} \exp\left[-\frac{x^2}{4a_1^2(t-\tau)}\right] d\tau + D_1, \tag{7}$$

$$u_2(x,\ t) = b_2 t + A_2 \operatorname{erf}\left(\frac{x}{2a_2\sqrt{t}}\right) + B_2\left(t + \frac{x^2}{2a_2^2}\right) + C_2 \int_0^t \frac{\sqrt{\tau}}{(t-\tau)^{1/2}} \exp\left[-\frac{x^2}{4a_2^2(t-\tau)}\right] d\tau + D_2, \tag{8}$$

where A_i, B_i, C_i, and D_i are arbitrary constants.

It is easy to see that the first term in (7) and (8) satisfies the inhomogeneous equations (1), (2), the second, third and fifth terms satisfy the corresponding homogeneous equations, and the fourth term, as the thermal potential of a simple layer, also satisfied the homogeneous equations (1). The arbitrary constants A_i, B_i, C_i, and D_i should be determined from the conditions (3)-(6). For this we shall first transform the thermal potential of a simple layer. Let

$$y = h_i \sqrt{\tau/(t-\tau)}, \quad h_i = |x|/2a_i\sqrt{t}\ . \tag{9}$$

Then

$$\int_0^t \frac{\sqrt{\tau}}{(t-\tau)^{1/2}} \exp\left[-\frac{x^2}{4a_i^2(t-\tau)}\right] d\tau = t\,\varkappa(h_i), \tag{10}$$

$$\varkappa(h_i) = 2h_i \exp(-h_i^2) \int_0^\infty \frac{y^2 \exp(-y^2)}{(y^2+h_i^2)^2}\, dy. \tag{11}$$

But it is well known [2] that

$$\int_0^\infty \frac{y^2 \exp(-y^2)}{(y^2+h_i^2)^2} dy = \frac{\pi}{2} \exp(h_i^2)\left(\frac{1}{h_i} + h_i\right) \operatorname{erfc} h_i - \frac{\sqrt{\pi}}{2}\ , \tag{12}$$

and thus

$$\varkappa(h_i) = \frac{\pi}{2}(1+2h_i^2) \operatorname{erfc} h_i - \sqrt{\pi}\, h_i \exp(-h_i^2). \tag{13}$$

If $x = -\alpha_1\sqrt{t}$ or $x = \alpha_2\sqrt{t}$, then $h_i = \alpha_i/2a_i$. Consequently,

$$\varkappa\left(\frac{\alpha_i}{2a_i}\right) = \varkappa_i = \frac{\pi}{2}\left(1 + \frac{\alpha_i^2}{2a_i^2}\right) \operatorname{erfc}\left(\frac{\alpha_i}{2a_i}\right) - \sqrt{\pi}\,\frac{\alpha_i}{2a_i} \exp\left(-\frac{\alpha_i^2}{4a_i^2}\right), \tag{14}$$

and if $x = 0$, then

$$\varkappa(0) = \pi/2. \tag{15}$$

Determination of the Coefficients

Now it remains to choose the arbitrary constants A_i, B_i, C_i, and D_i so as to satisfy conditions (3)-(6). Condition (3) yields

$$b_1 t + A_1 \operatorname{erf}\left(-\frac{\alpha_1}{2a_1}\right) + B_1\left(t + \frac{\alpha_1^2}{2a_1^2}\, t\right) + C_1 \varkappa_1 t + D_1 = u_{n1}.$$

Equating the coefficients of the right-hand and left-hand sides, we obtain

$$\begin{gathered} b_1 + B_1(1 + \alpha_1^2/2a_1^2) + C_1\varkappa_1 = 0, \\ A_1 \operatorname{erf}(-\alpha_1/2a_1) + D_1 = u_{n1}. \end{gathered} \tag{16}$$

In an analogous way, equation (4) yields

$$\begin{gathered} b_2 + B_2(1 + \alpha_2^2/2a_2^2) + C_2\varkappa_2 = 0, \\ A_2 \operatorname{erf}(\alpha_2/2a_2) + D_2 = u_{n2}. \end{gathered} \tag{17}$$

Applying condition (5), we obtain

$$b_1 t + B_1 t + C_1 \frac{\pi}{2} t + D_1 = b_2 t + B_2 t + C_2 \frac{\pi}{2} t + D_2,$$

and hence

$$\begin{gathered} b_1 + B_1 + \frac{\pi}{2} C_1 = b_2 + B_2 + \frac{\pi}{2} C_2, \\ D_1 = D_2. \end{gathered} \tag{18}$$

Taking into account that

$$\frac{\partial u_i}{\partial x}=A_i\frac{2\exp[-x^2/4a_i^2t]}{\sqrt{\pi}\,2a_i\sqrt{t}}+B_i\frac{x}{a_i^2}-C_1\int_0^t\frac{x\exp[-x^2/4a_i^2(t-\tau)]}{2a_i^2(t-\tau)^{3/2}}\sqrt{\tau}\,d\tau$$

and

$$\lim_{x\to\pm 0}\int_0^t\frac{x\sqrt{\tau}}{2a_i^2(t-\tau)^{3/2}}\exp\left[-\frac{x^2}{4a_i^2(t-\tau)}\right]d\tau=\pm\frac{\sqrt{\pi}}{a_i}\sqrt{t},$$

equation (6) gives

$$\lambda_1\left[\frac{A_1}{a_1\sqrt{\pi}\sqrt{t}}+C_1\frac{\sqrt{\pi}}{a_1}\sqrt{t}\right]=\lambda_2\left[\frac{A_2}{a_2\sqrt{\pi}\sqrt{t}}-C_2\frac{\sqrt{\pi}}{a_2}\sqrt{t}\right],$$

which determines the two conditions

$$\begin{gathered}\frac{\lambda_1}{a_1\sqrt{\pi}}A_1=\frac{\lambda_2}{a_2\sqrt{\pi}}A_2,\\ \lambda_1\frac{\sqrt{\pi}}{a_1}C_1=-\lambda_2\frac{\sqrt{\pi}}{a_2}C_2.\end{gathered}\tag{19}$$

The system of equations (16)-(19) splits into two independent systems:

$$\begin{gathered}(1+\alpha_1^2/2a_1^2)B_1+\varkappa_1C_1=-b_1,\\ (1+\alpha_2^2/2a_2^2)B_2+\varkappa_2C_2=-b_2,\\ B_1+\frac{\pi}{2}C_1-B_2-\frac{\pi}{2}C_2=b_2-b_1,\\ \frac{\lambda_1}{a_1}C_1+\frac{\lambda_2}{a_2}C_2=0;\end{gathered}\tag{20}$$

$$\begin{gathered}A_1\operatorname{erf}(-\alpha_1/2a_1)+D_1=u_{n1},\\ A_2\operatorname{erf}(\alpha_2/2a_2)+D_2=u_{n2},\\ D_1=D_2,\\ \frac{\lambda_1}{a_1}A_1=\frac{\lambda_2}{a_2}A_2\ .\end{gathered}\tag{21}$$

Solving the systems (20) and (21) we obtain

$$\begin{gathered}A_1=\frac{a_1\lambda_2(u_{n2}-u_{n1})}{a_1\lambda_2\operatorname{erf}(\alpha_1/2a_1)+a_2\lambda_1\operatorname{erf}(\alpha_2/2a_2)},\\ A_2=\frac{a_2\lambda_1(u_{n2}-u_{n1})}{a_1\lambda_2\operatorname{erf}(\alpha_1/2a_1)+a_2\lambda_1\operatorname{erf}(\alpha_2/2a_2)},\\ D_1=D_2=\frac{u_{n2}a_1\lambda_2\operatorname{erf}(\alpha_1/2a_1)+u_{n1}a_2\lambda_1\operatorname{erf}(\alpha_2/2a_2)}{a_1\lambda_2\operatorname{erf}(\alpha_1/2a_1)+a_2\lambda_1\operatorname{erf}(\alpha_2/2a_2)},\\ C_1=\frac{2a_1\lambda_2[(b_2-b_1)\omega_1\omega_2+b_1\omega_2-b_2\omega_1]}{a_1\lambda_2\omega_2(\pi\omega_1-2\varkappa_1)+a_2\lambda_1\omega_1(\pi\omega_2-2\varkappa_2)},\\ C_2=\frac{2a_2\lambda_1[(b_1-b_2)\omega_1\omega_2+b_2\omega_1-b_1\omega_2]}{a_1\lambda_2\omega_2(\pi\omega_1-2\varkappa_1)+a_2\lambda_1\omega_1(\pi\omega_2-2\varkappa_2)},\\ B_1=-\frac{\varkappa_1}{\omega_1}C_1-\frac{b_1}{\omega_1},\ B_2=-\frac{\varkappa_2}{\omega_2}C_2-\frac{b_2}{\omega_2},\end{gathered}\tag{22}$$

where

$$\omega_i = 1 + \alpha_i^2 / 2a_i^2 . \tag{23}$$

As a practical application of these results, we shall consider the bridge erosion of electric contacts.

During the opening of electric contacts there forms a bridge of molten electrode metal. In the absence of an arc discharge, this liquid bridge results in the transfer of material from one electrode to the other. In the case of dc relays carrying a light load, the transfer of material by means of the liquid bridge is a most deleterious effect, which results in the premature wear of the contacts and in unreliable operation with small contact forces.

It is known that the temperature fields in the liquid cylinder and in the adjoining solid contact electrodes are one-dimensional. Therefore we shall choose the x axis to coincide with the axis of the cylinder, and we shall choose the origin to be at the point of contact of the two electrodes.

The problem of the advancement of the liquid phase into the solid electrodes can be reduced to the analogous problem of freezing. Using the approximate method of solution of such problems [3], the advancement of the liquid phase into the solid phase can be represented by the equation

$$x_1 = -\alpha_1 \sqrt{t}, \quad x_2 = \alpha_2 \sqrt{t}, \tag{24}$$

where α_i are coefficients determined from the equations

$$\frac{\lambda_i (u_b - u_m)}{\alpha_i} - \frac{2 (u_m - u_{0i})}{a \sqrt{\pi}} = Q \gamma \alpha_i \quad (i = 1, 2). \tag{25}$$

The length of the liquid bridge at a given instant of time can be determined from

$$l = (\alpha_1 + \alpha_2) \sqrt{t} + Vt. \tag{26}$$

In most contact materials the value of α_i is of the same order as the speed of separation of the contacts V. Consequently, for small values of t, $\alpha_i \sqrt{t} \gg Vt$, so that the value Vt can be neglected.

The mean diameter of the liquid bridge is given by the equation

$$d_m = \mu d_0 l. \tag{27}$$

When the electric contacts are of the same material, then $a_1^2 = a_2^2 = a^2$, $\lambda_1 = \lambda_2 = \lambda$, $u_{m1} = u_{m2} = u_m$, $b_1 = b_2 = b = \rho I^2 / \gamma cF$. But due to the tunnel effect the temperature of the contact surface of the anode is higher than that of the cathode and, consequently, $\alpha_1 > \alpha_2$.

Thus, in this case the coefficients in (7) and (8) are, in accordance with (22),

$$A_1 = A_2 = 0, \; D_1 = D_2 = u_m,$$

$$C_1 = C_2 = \frac{b (\omega_2 - \omega_1)}{\pi \omega_1 \omega_2 - \varkappa_1 \omega_2 - \varkappa_2 \omega_1}, \tag{28}$$

$$B_1 = -\frac{\varkappa_1}{\omega_1} C_1 - \frac{b}{\omega_1}, \quad B_2 = -\frac{\varkappa_2}{\omega_2} C_2 - \frac{b}{\omega_2}.$$

Actually, we are interested in the volume of material transferred during one opening of the circuit $v = x_0 F$. The determination of x_0 and the corresponding time t_0 reduces to a problem of finding an extremum of an implicit function, i.e., to determining the root of the system

$$u(x_0, t_0) = u_b, \qquad \frac{\partial u(x_0, t_0)}{\partial x} = 0. \tag{29}$$

For the left-hand side of the bridge these equations become

$$bt_{01}+B_1\left(t_{01}+\frac{x_{01}^2}{2a^2}\right)+C_1\int\limits_0^{t_{01}}\frac{\sqrt{\tau}}{(t_{01}-\tau)^{1/2}}\exp\left[-\frac{x_{01}^2}{4a^2(t_{01}-\tau)}\right]d\tau+D_1=u_b,$$

$$B_1\frac{x_{01}}{a^2}-C_1\int\limits_0^{t_{01}}\frac{x_{01}\sqrt{\tau}}{2a^2(t_{01}-\tau)^{3/2}}\exp\left[-\frac{x_{01}^2}{4a^2(t_{01}-\tau)}\right]d\tau=0. \quad (30)$$

Using equations (9), (10) and (11), we obtain

$$bt_{01}+B_1(t_{01}+2h_1t_{01})+C_1t_{01}\varkappa(h_1)=u_b-D_1,$$

$$-B_1\frac{2h_1\sqrt{t_{01}}}{a}-C_1\frac{t_{01}}{2a\sqrt{t_{01}}}\varkappa'(h_1)=0, \quad (31)$$

where

$$\varkappa'(h_1)=2\sqrt{\pi}\,(\sqrt{\pi}h_1\operatorname{erfc}h_1-\exp(-h_1^2)).$$

Hence

$$t_{01}=(u_\kappa-u_m)[b+B_1(1+2h_1^2)+C_1\varkappa(h_1)]^{-1}, \quad (32)$$

$$h_1=\frac{C_1\sqrt{\pi}}{2B_1}(\exp(-h_1^2)\sqrt{\pi}h_1\operatorname{erfc}h_1). \quad (33)$$

In an analogous way we obtain the equations for the right-hand side

$$t_{02}=(u_b-u_m)[b+B_2(1+2h_2^2)+C_2\varkappa(h_2)]^{-1}, \quad (34)$$

$$h_2=(\exp[-h_2^2]+\sqrt{\pi}h_2\operatorname{erfc}h_2)C_2\sqrt{\pi}/2B_2. \quad (35)$$

It is easy to see that C_1/B_1 and C_2/B_2 have different signs, so that equation (35) has one root, and (33) has no roots. Physically, this means that the bridge can burn at one point only, either at left or at right.

Substituting the value of h_1 (or h_2) into (32) [or (34)], we determine the time of burnout of the bridge t_{01} (or t_{02}), and then find the coordinate of the point of burnout according to $x_{01}=-2ah_1\sqrt{t_{01}}$ or $x_{02}=2ah_2\sqrt{t_{02}}$.

The volume of the material transferred during one opening of the circuit is given by the equation $v=2aFh\sqrt{t_0}$.

Numerical Example

We shall calculate the bridge erosion of platinum electrodes due to the opening of a circuit carrying one ampere of current. The thermophysical constants of platinum are as follows: $\rho_{20}=11\cdot10^{-6}$ ohm · cm, $a=0.5$ cm/sec$^{1/2}$, $\lambda=0.7$ W/cm · deg, $u_b=4663$°K, $u_m=2026$°K, $\sigma=10^{-8}$ ohm · cm^2, $\gamma=21.4$ g/cm^3, $\gamma c=2.8$ J/cm^3 · deg.

The diameter of the liquid bridge in platinum is given by the expression

$$d_m=\mu d_0=17{,}8\,\mu\cdot10^{-5}\text{ cm}.$$

The temperature of the cathode u_2 for a contact-surface radius $r=8.9\,\mu\cdot10^{-5}$ cm is given by the expression [3]

$$u_2|_{t=0}=5\rho_{20}I^2/32\lambda r;\ u_2=u_{02}=583°\text{ K}.$$

The mean temperature in the melting zone of the anode is

$$u_{01}=(u_{02}+u_m)/2;\ u_{01}=1304°\text{ K}.$$

Substituting u_{01} or u_{02} in (25), we obtain α_1 or α_2, respectively

$$\alpha_1=0.68\text{ cm/sec}^{1/2},\ \alpha_2=0.53\text{ cm/sec}^{1/2}.$$

For the determination of the coefficient b we have used the resistivity $\rho = 10\rho_{20}$, which in the case of platinum corresponds to the boiling point:

$$b = 10\,\rho_{20}\,I^2/\mu^4\gamma\, cF^2,\ \ b = 1/\mu^4 \cdot 6.47 \cdot 10^{10}\ \text{deg/sec}.$$

The values ω_1 and ω_2 are given by

$$\omega_1 = 1 + \alpha_1^2/2a^2,\ \ \omega_1 = 1.925;$$

$$\omega_2 = 1 + \alpha_2^2/2a^2,\ \ \omega_2 = 1.56.$$

Equation (13) yields

$$\varkappa_1 = 0.256,\ \ \varkappa_2 = 0.385.$$

Equation (28) gives

$$C_1 = -\frac{1}{\mu^4}0.28\cdot 10^{10}\ \text{deg/sec},\ \ C_2 = \frac{1}{\mu^4}0.28\cdot 10^{10}\ \text{deg/sec}$$

and

$$B_1 = -\frac{1}{\mu^4}3.32\cdot 10^{10}\ \text{deg/sec},\qquad B_2 = -\frac{1}{\mu^4}4.21\cdot 10^{10}\ \text{deg/sec}.$$

Taking into account that $C_1/B_1 > 0$, equation (33) yields

$$h_1 = 0.075.$$

Equation (32) gives

$$t_{01} = \mu^4 \cdot 0.972 \cdot 10^{-7}\ \text{sec}.$$

Using the value $\mu = 0.56$ [4], we find the volume of metal transferred during one opening of the circuit:

$$v = 5.77 \cdot 10^{-14}\ \text{cm}^3\ \text{per opening}.$$

Calculations for gold, silver and palladium give $v \cdot 10^{14} = 4.49$, 7.65, and 6.31, respectively. The amount of metal transferred at different values of the current was measured with great precision by Lander and Germer [5], using shadowgraph and microscopic methods. Curves drawn through their experimental points represent, in the case of platinum, silver, gold, and palladium, the equation

$$v = 6\cdot 10^{-14}\ I^3\ \text{cm}^3\ \text{per opening}.$$

Thus, our results, obtained analytically, are in excellent agreement with experimental data.

NOTATION

u_{01}, u_{02} – initial temperatures of contact surfaces of anode and cathode; u_b, u_m – boiling and melting temperature, respectively; μ – ratio of smallest diameter in central part of bridge to the maximum diameter at the anode $d_M = d_0 I$; d_0 – proportionality factor; I – current [4]; γ – specific gravity; c – specific heat; ρ – resistivity; F – area of mean cross-section of bridge; x_0 – coordinate of the point at which the temperature first reaches the boiling point u_b.

REFERENCES

1. A. N. Tikhonov and A. A. Samarskii, Equations of Mathematical Physics [in Russian], Fizmatgiz, 1951.
2. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series and Products [in Russian], Fizmatgiz, 1962.
3. A. V. Lykov, Theory of Heat Transfer [in Russian] Gostekhizdat, 1952.
4. Davidson, Br. J. Appl. Phys., 5, no. 5, 1954.
5. J. J. Lander and L. H. Germer, J. Appl. Phys., 19, no. 10, 1948.

8 July 1964

Lenin Polytechnic Institute, Khar'kov